

Iterating free-field AdS/CFT: higher spin partition function relations

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ABSTRACT: We find a simple relation between a free higher spin partition function on thermal quotient of AdS_{d+1} and the partition function of the associated d -dimensional conformal higher spin field defined on thermal quotient of AdS_d . Starting with a conformal higher spin field defined in AdS_d one may also associate to it another conformal field in $d - 1$ dimensions, thus "iterating" AdS/CFT. We observe that in the case of $d = 4$ this iteration leads to a "trivial" 3d higher spin conformal theory with parity-even non-local action: it describes zero total number of dynamical degrees of freedom and the corresponding partition function is equal to 1.

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1 Introduction

"Kinematical" AdS/CFT correspondence relates a field ϕ in AdS_{d+1} (e.g., with standard 2-derivative action with some mass parameter M^2 or associated dimension Δ) to a conformal field ϕ at the boundary $\mathcal{M}^d = \partial(\text{AdS}_{d+1})$ with canonical dimension $\Delta^- = d - \Delta$. The value of AdS mass parameter and thus of Δ^- determines the number of derivatives in the kinetic term in the action for ϕ :

$$S_d = \int d^d x \, \phi \partial^k \phi, \quad k = d - 2\Delta^- = 2\Delta - d. \quad (1.1)$$

For example, a massless totally symmetric higher spin field ϕ_s in AdS_{d+1} is associated to a conformal higher spin field ϕ_s with the action $S_d = \int d^d x \, \phi_s P_s \partial^{2s+d-4} \phi_s$ where P_s is

traceless transverse projector (see [1] for a review and refs.). From the standard AdS/CFT perspective a massless AdS field φ_s is a counterpart of a bilinear conserved current J_s of a free (e.g., scalar Φ) boundary CFT $_d$ while ϕ_s is associated with a shadow field or a source for J_s ; thus the action for ϕ_s may be interpreted as an "induced" action found upon integrating out Φ coupled to ϕ_s via $J_s(\Phi)$.

There are other relations between the two free theories $S_{d+1}(\varphi)$ and $S_d(\phi)$ in $d+1$ and d dimensions beyond just kinematic $SO(2, d)$ representation theory correspondence. First, the AdS_{d+1} action for φ evaluated on the solution of the Dirichlet problem $\varphi|_{\partial} = \phi$ gives an "induced" action for ϕ . For even d and, e.g., for a massless field φ the AdS_{d+1} action contains a logarithmically divergent local term which is identified with a local action for ϕ . For odd d one gets, in general, a non-local action as the power k in the kinetic operator in (1.1) may be half-integer or negative. In addition to this tree-level relation there is also a 1-loop one – between ratio of partition functions for a higher spin (HS) field φ in AdS_{d+1} with D and N (or $+$ and $-$) boundary conditions and for the dual conformal field (CF) ϕ at the boundary:

$$\frac{Z_{\text{HS}}^-}{Z_{\text{HS}}^+} \Big|_{\text{AdS}_{d+1}} = Z_{\text{CF}} \Big|_{\partial(\text{AdS}_{d+1})} . \quad (1.2)$$

This relation is true, e.g., for global AdS_{d+1} with boundary S^d and also for a thermal quotient of AdS_{d+1} with boundary $S^1_{\beta} \times S^{d-1}$. In the latter case (1.2) translates into a relation between one-particle partition functions \mathcal{Z} as functions of the $q = e^{-\beta}$ variable¹

$$\mathcal{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{CF}}(S^1 \times S^{d-1}; q) , \quad (1.3)$$

$$Z = \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{Z}(q^n) , \quad (1.4)$$

$$\mathcal{Z}^-(\text{AdS}_{d+1}; q) = \tilde{\mathcal{Z}}^-(\text{AdS}_{d+1}; q) + \sigma_{d+1}(q) , \quad (1.5)$$

$$\tilde{\mathcal{Z}}^-(\text{AdS}_{d+1}; q) \equiv (-1)^d \mathcal{Z}^+(\text{AdS}_{d+1}; q^{-1}) . \quad (1.6)$$

Eq. (1.3) may be interpreted in terms of counting of operators in the boundary CFT or as a group-theoretic relation for characters of the conformal group. More generally, (1.2) is expected to be true even for asymptotically AdS space and its generic curved boundary (provided the corresponding $d+1$ and d dimensional theories can be consistently defined) and should thus provide, in particular, an AdS theory based way to compute not only the conformal anomaly a -coefficients [2] but also the c -coefficients [3].

Having identified a conformal field ϕ in \mathbb{R}^d associated to a field φ in AdS_{d+1} we may attempt to repeat this step one more time. Namely, we may first define this ϕ not on \mathbb{R}^d (or S^d or $S^1 \times S^{d-1}$) but on AdS_d and then associate to it *another* conformal field $\hat{\phi}$ in $d-1$

¹Here $\sigma_d(q)$ is a finite polynomial in $q + q^{-1}$ that represents contribution of finite number of "zero" modes related to gauge invariance of the conformal (shadow) field [2, 1].

dimensions. We will then have the following dimensional ($d + 1 \rightarrow d \rightarrow d - 1$) digression²

$$\varphi(\text{AdS}_{d+1}) \rightarrow \varphi(\partial \text{AdS}_{d+1} \propto \text{AdS}_d) \rightarrow \widehat{\varphi}(\partial \text{AdS}_d). \quad (1.7)$$

If φ is a gauge field with 2-derivative action in AdS_{d+1} , then φ is also single (gauge) conformal field with, in general, higher derivative action. The latter can be represented as a collection of 2nd-derivative fields in AdS_d and hence $\widehat{\varphi}$ in $d - 1$ dimensions will be given by set of several conformal fields, each being dual to an individual 2nd-derivative field in d dimensions.

Our aim below will be to explore some implications of this "iterated" AdS/CFT correspondence (1.7) at the level of relations between partition functions like (1.2) and (1.3). We shall find that for a generic higher spin field (HS) in AdS_{d+1} and its dual conformal field (CF) in d dimensions one gets also

$$\mathcal{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q), \quad (1.8)$$

where \mathcal{Z}^- may be replaced by $\widetilde{\mathcal{Z}}^-$ in (1.6) as one finds also that the σ terms in (1.5) match, $\sigma_{\text{HS}, d+1} = \sigma_{\text{CF}, d}$. Eq. (1.8) follows from (1.3) and

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) = \mathcal{Z}_{\text{CF}}(S^1 \times S^{d-1}; q), \quad (1.9)$$

which may be related to the fact that AdS_d is conformal to half of $\mathbb{R} \times S^{d-1}$ so that the respective partition functions are related provided one sums over the two possible boundary conditions at the boundary of AdS_d . Applying (1.3) to CF in AdS_d and its counterpart conformal field $\widehat{\text{CF}}$ in $d - 1$ dimensions (cf. (1.7)) we get also

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) - \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) = \mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^{d-2}; q). \quad (1.10)$$

We shall find that the case of $d = 4$ is special: starting with a HS field in AdS_5 , the resulting 3d conformal theory represented by $\widehat{\varphi}$ is effectively "topological", having zero number of dynamical d.o.f. and trivial partition function. This may be related to equivalence of \pm modes with non-zero spins in AdS_4 [4], implying

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_4; q) = \mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q) \rightarrow \mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^2; q) = 0. \quad (1.11)$$

Very loosely, this may be interpreted as a version of the "boundary of boundary = 0" relation, or as " $(\text{AdS/CFT})^2 = 0$ ".³

²For standard global AdS_{d+1} we have $\partial \text{AdS}_{d+1} = \mathbb{R} \times S^{d-1}$. This space is equivalent to two copies of AdS_d glued along their boundary identified with the equator of S^{d-1} . The middle step in (1.7) means that we start with the conformal action on $\partial \text{AdS}_{d+1}$ and then translate it into AdS_d (taking also into account the freedom in choice of boundary conditions, see below).

³Let us note that our interpretation and examples will be different from previous discussions of "sequential" AdS/CFT like $\text{AdS}_4/\text{CF}_3 \rightarrow \text{AdS}_3/\text{CF}_2$ in [5, 6] (for related work discussing AdS_d foliations of AdS_{d+1} see also [7–11]). In particular, in contrast to [5] the 3d conformal higher spin theory that will naturally appear in our context is not of local Chern-Simons type but has parity-even non-local action. Let us also mention for completeness that discussions of dimensional reduction from AdS_{d+1} to AdS_d appeared in [12, 13].

We shall start in section 2 with a review of some general definitions and relations. Then in section 3 we shall demonstrate the validity of (3.1) on several examples, in particular, for massless higher spin fields in AdS_{d+1} related to conformal higher spin fields in AdS_d .

In section 4 we shall first analyse the detailed structure of the relation (3.1) on the example of the totally symmetric field in AdS_5 with generic mass parameter and mention its possible group-theoretic interpretation and then justify the $\mathcal{Z}^- = \mathcal{Z}^+$ equality in (1.11). We shall then discuss in detail the corresponding 3d conformal theory with non-local linearized action describing total of zero degrees of freedom and leading to trivial partition function. We shall use spin 1 Maxwell and spin 2 conformal graviton fields as examples.

Section 5 will contain some concluding remarks. In Appendix A we shall discuss the algebraic structure of the partition functions appearing in (3.1) and then in Appendix B argue for the equality of the corresponding σ -terms in (1.5). In Appendices C, D and E the relation (3.1) will be further illustrated on the examples of conformal higher derivative scalars, fermionic conformal higher spin fields and conformal antisymmetric tensor field in 4d.

2 Some general relations

Let us consider a conformally invariant action in AdS_d . This space is conformally equivalent to one half of static Einstein universe $S^1 \times S^{d-1}$, with the boundary of AdS_d being mapped to the equator of S^{d-1} [4, 14, 15]. One can consider the single particle partition function $\mathcal{Z}(\text{AdS}_d; q)$ on thermal AdS_d where we identify $t \sim t + \beta$. This can be compared with the partition function in Einstein universe $\mathcal{Z}(S^1 \times S^{d-1}; q)$ where S^1 is the thermal circle with length β .

The calculation of total partition function $Z(\text{AdS}_d; q)$ (and thus of $\mathcal{Z}(\text{AdS}_d; q)$) is straightforward assuming that the kinetic operator of a conformal field factorises, i.e. the action in AdS_d can be written a sum of 2nd-derivative terms (as, e.g., in [16]). For example, let us consider

$$\log Z(\text{AdS}_d) = -\frac{1}{2} \sum_{i=1}^N n_i \log \det \hat{\Delta}_{s_i \perp}(M_i^2), \quad \hat{\Delta}_{s \perp}(M^2) \equiv (-\nabla^2 - M^2)_{s \perp} \quad (2.1)$$

where $\hat{\Delta}_{s \perp}$ is defined on transverse traceless symmetric tensors of rank s ⁴, and the integers n_i are field multiplicities positive for physical fields and negative for ghost fields. For each operator in (2.1) the value of mass term then determines possible ground state energies Δ_d^\pm that are solutions of the quadratic equation [17–19]

$$\Delta_d^\pm (\Delta_d^\pm - d + 1) - s = -M^2, \quad \Delta_d^- = d - 1 - \Delta_d^+, \quad \Delta_d^- \leq \Delta_d^+, \quad (2.2)$$

and are associated with classical solutions of the wave equation $\hat{\Delta}_{s \perp}(M^2) \varphi_{s \perp} = 0$ with two different boundary conditions. Taking the thermal quotient of AdS_d , we then get

⁴In general, we define $\hat{\Delta}_{s \perp}(M^2) = (-\nabla^2 + M^2 \epsilon)_{s \perp}$, where $\epsilon = -1$ for AdS_d and $\epsilon = +1$ for S^d (here we set the curvature scale to 1).

from (2.1) the following two possibilities for the corresponding single particle partition function ($q = e^{-\beta}$)

$$\mathcal{Z}^+(\text{AdS}_d; q) = \sum_{i=1}^N n_i g_{s_i}^{(d)} \frac{q^{\Delta_{d,i}^+}}{(1-q)^{d-1}}, \quad \tilde{\mathcal{Z}}^-(\text{AdS}_d; q) = \sum_{i=1}^N n_i g_{s_i}^{(d)} \frac{q^{\Delta_{d,i}^-}}{(1-q)^{d-1}}. \quad (2.3)$$

In (2.3) $g_s^{(d)}$ is the multiplicity that counts the number of off-shell degrees of freedom⁵

$$g_s^{(d)} = (2s + d - 3) \frac{(s + d - 4)!}{(d - 3)!s!}. \quad (2.4)$$

Using that $\Delta^- = d - 1 - \Delta^+$ we find

$$\tilde{\mathcal{Z}}^-(\text{AdS}_d; q) = (-1)^{d-1} \mathcal{Z}^+(\text{AdS}_d; q^{-1}). \quad (2.5)$$

In the presence of gauge invariance the proper $\mathcal{Z}^-(\text{AdS}_d; q)$ partition function differs from $\tilde{\mathcal{Z}}^-(\text{AdS}_d; q)$

$$\mathcal{Z}_{\text{HS}}^-(\text{AdS}_d; q) = \tilde{\mathcal{Z}}_{\text{HS}}^-(\text{AdS}_d; q) + \sigma_d(q), \quad (2.6)$$

where $\sigma(q)$ is a finite polynomial in $q + q^{-1}$ related to missing gauge transformations discussed in [1].

The calculation of $\mathcal{Z}(S^1 \times S^{d-1}; q)$ on the Einstein universe background is a priori unrelated to the one on AdS_d . If the action on generic \mathcal{M}^d is known, one may just specialize it to $S^1 \times S^{d-1}$, factorize the kinetic operator and use the methods discussed in [1].⁶ Alternatively, one can make use of the conformal map to flat space \mathbb{R}^d ("radial quantization") and use flat space operator counting techniques.

At the same time, $\mathcal{Z}(S^1 \times S^{d-1}; q)$ can also be computed starting with the dual theory in AdS_{d+1} (cf. (1.3)). If $S^1 \times S^{d-1}$ is interpreted as the boundary of AdS_{d+1} the corresponding conformally invariant action on $S^1 \times S^{d-1}$ can be interpreted as "induced" from an action of a dual field in the bulk (see, e.g., [2, 1] and refs. there). Let us call a generic tensor bulk field a "higher spin" (HS) one; this name will include the cases of a massive or partially massless or exactly massless higher spin fields in AdS_{d+1} . The dual conformal field at the boundary will be denoted as CF. Then [1]

$$\mathcal{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{CF}}(S^1 \times S^{d-1}; q). \quad (2.7)$$

In the case of massless higher spin (MHS) field in AdS_{d+1} having maximal gauge invariance the associated conformal field at the boundary is conformal higher spin (CHS) one and σ_{d+1} in (2.6) is non-trivial [1].⁷

⁵Special cases are $g_s^{(4)} = 2s + 1$, $g_s^{(6)} = \frac{1}{6}(s + 1)(s + 2)(2s + 3)$.

⁶If one knows the set of masses M^2 in (2.1) for an action on AdS_d , this is not enough to compute the partition function for the same theory on $S^1 \times S^{d-1}$. The reason is that M^2 values come from the specialization to AdS_d of the action on a generic curved background \mathcal{M}^d where certain combinations of curvature tensor terms lead to mass terms. Specification of this action to $S^1 \times S^{d-1}$ will then lead to different kinetic term structures, i.e. to different mass terms in the corresponding 2nd-order operators.

⁷In addition to the quantum "one-loop" relation (2.7) the quadratic actions for HS and CF have also classical relation: evaluating the action of HS field in AdS_{d+1} on the solution with boundary data being equal to CF field one gets the action of the CF field as an "induced" one. In even d case the local CF action is the coefficient of the leading logarithmic IR divergence while in odd d case it is finite but non-local.

3 Partition functions on AdS_{d+1} and AdS_d

Let us now propose and check on several examples a general relation between partition functions of higher spin field in AdS_{d+1} and associated conformal field originally "induced" on $\partial\text{AdS}_{d+1} = \mathbb{R} \times S^{d-1}$ or S^d but that can then be also defined on AdS_d . This relation is (1.8) that we rewrite here for the reader's convenience

$$\mathcal{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q). \quad (3.1)$$

Heuristically, the relation (3.1) may be motivated as follows. AdS_d is conformal to half of the Einstein universe $S^1 \times S^{d-1}$ with two possible choices of the boundary conditions at the equator; thus defining the partition function on $S^1 \times S^{d-1}$ in terms of AdS_d one we may need to sum over the two boundary condition choices,

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) = \mathcal{Z}_{\text{CF}}(S^1 \times S^{d-1}; q). \quad (3.2)$$

Combining this with (2.7) then gives (3.5).

Note that starting with a CF field in AdS_d we may also associate to it another conformal field $\widehat{\text{CF}}$ at the $d-1$ boundary and then the analog of (2.7) will read

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q) - \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) = \mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^{d-2}; q). \quad (3.3)$$

Furthermore, the $\sigma(q)$ terms in (2.6) for $\mathcal{Z}_{\text{HS}}^-(\text{AdS}_{d+1}; q)$ and $\mathcal{Z}_{\text{CF}}^-(\text{AdS}_d; q)$ appear to match (see Appendix B)

$$\sigma_{\text{HS}, d+1}(q) = \sigma_{\text{CF}, d}(q) \quad (3.4)$$

so that (3.1) may be written also as

$$\widetilde{\mathcal{Z}}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \widetilde{\mathcal{Z}}_{\text{CF}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q). \quad (3.5)$$

Using (2.5) this can be put also in the following more symmetric form

$$-\mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) + (-1)^d \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q^{-1}) = \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) - (-1)^d \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q^{-1}) \quad (3.6)$$

Below we will demonstrate the validity of (3.1), (3.5) and (3.2) on several examples of conformal fields (for some consequences of (3.5) see also Appendix A).

3.1 Conformal scalar

Let us start with the case of a particular scalar field in AdS_{d+1} with the mass term $M^2 = \frac{1}{4}d^2 + 1$, i.e. with $\Delta_{d+1}^\pm = \frac{1}{2}(d \pm 2)$ (cf. (2.2)). The corresponding partition (2.3) function is

$$\mathcal{Z}_0^\pm(\text{AdS}_{d+1}; q) = \frac{q^{\frac{1}{2}(d \pm 2)}}{(1 - q)^d}. \quad (3.7)$$

This scalar in AdS_{d+1} "induces" a spin 0 field φ at the boundary with canonical dimension $= \Delta_{d+1}^- = \frac{1}{2}(d - 2)$, i.e. which thus represents a conformally coupled scalar. The corresponding kinetic operator in a curved d -dimensional space specified to the case of the unit-scale AdS_d (with $R = -d(d - 1)$) is

$$-\nabla^2 + \frac{d-2}{4(d-1)}R = -\nabla^2 - \frac{1}{4}d(d-2). \quad (3.8)$$

Thus defining φ on AdS_d we find that the mass term (cf. (2.1)) is $M^2 = \frac{1}{4}d(d-2)$ and thus from (2.2)

$$\Delta_d^+ = \frac{d}{2}, \quad \Delta_d^- = \frac{d-2}{2}. \quad (3.9)$$

From (2.3) the partition functions corresponding to this conformal scalar (cs) are then

$$\mathcal{Z}_{\text{cs}}^+(\text{AdS}_d; q) = \frac{q^{d/2}}{(1-q)^{d-1}}, \quad \mathcal{Z}_{\text{cs}}^-(\text{AdS}_d; q) = (-1)^{d+1} \mathcal{Z}_{\text{cs}}^+(\text{AdS}_d; q^{-1}), \quad (3.10)$$

Comparing to (3.7) one can then check that

$$\mathcal{Z}_0^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_0^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{cs}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{cs}}^+(\text{AdS}_d; q), \quad (3.11)$$

which is a particular spin 0 case of (3.1).

To demonstrate (3.2) we recall that the partition function on $S^1 \times S^{d-1}$ can be found, e.g., by the operator counting method. For a scalar φ with canonical dimension $\frac{1}{2}(d-2)$ and equations of motion $\partial^2 \varphi = 0$ that gives

$$\mathcal{Z}_{\text{cs}}(S^1 \times S^{d-1}; q) = \frac{q^{\frac{1}{2}(d-2)} - q^{\frac{1}{2}(d+2)}}{(1-q)^d}. \quad (3.12)$$

Then one can check that (2.7) is satisfied (there is no gauge invariance so $\sigma(q) = 0$ in (2.6)). As a result, we verify a special case of (3.2)

$$\mathcal{Z}_{\text{cs}}(S^1 \times S^{d-1}; q) = \mathcal{Z}_{\text{cs}}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{cs}}^+(\text{AdS}_d; q). \quad (3.13)$$

As already mentioned above, this relation means that one needs to sum over both \pm scalar modes in AdS_d in order to match the conformal scalar partition function on $S^1 \times S^{d-1}$ space which is conformally equivalent to a double copy of AdS_d .

The above discussion can be extended to higher derivative GJMS conformal scalars with higher-derivative kinetic operators, see Appendix C, again verifying the general relations (3.1)–(3.5).

3.2 Conformal higher spins

Let us now consider a totally symmetric spin s CHS field in d dimensions. The CHS theory defined on AdS_d has the following partition function [20, 21]

$$Z_{\text{CHS},s}(\text{AdS}_d) = \prod_{k=0}^{s-1} \left[\frac{\det \widehat{\Delta}_{k\perp}(M_{k,s}^2)}{\det \widehat{\Delta}_{s\perp}(M_{s,k}^2)} \right]^{1/2} \prod_{k'=-\frac{1}{2}(d-4)}^{-1} \left[\frac{1}{\det \widehat{\Delta}_{s\perp}(M_{s,k'}^2)} \right]^{1/2}, \quad (3.14)$$

$$M_{n,k}^2 \equiv n - (k-1)(k+d-2). \quad (3.15)$$

For each determinant here (cf. (2.1)) we may then compute the corresponding contribution to the one-particle partition function using the general relations (2.2),(2.3). As a result, we get

$$\mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_d; q) = \frac{1}{(1-q)^{d-1}} \left\{ \sum_{k=0}^{s-1} \left[g_s^{(d)} q^{d+k-2} - g_k^{(d)} q^{d+s-2} \right] + \sum_{k'=-\frac{1}{2}(d-4)}^{-1} g_s^{(d)} q^{d+k'-2} \right\},$$

$$\tilde{\mathcal{Z}}_{\text{CHS},s}^-(\text{AdS}_d; q) = (-1)^{d-1} \mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_d; q^{-1}). \quad (3.16)$$

Doing the sum, we find

$$\begin{aligned} \mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_d; q) = \frac{\Gamma(d+s-3)}{\Gamma(d-1)\Gamma(s+1)} \frac{q^{\frac{d}{2}-2}}{(1-q)^d} \Big[(d-2)(d+2s-3)q^2 \\ - (d+s-3)(d+2s-2)q^{\frac{d}{2}+s} + s(d+2s-4)q^{\frac{d}{2}+s+1} \Big]. \end{aligned} \quad (3.17)$$

In the special case of the $s = 2$ CHS field or, equivalently, of Weyl gravity on thermal quotient of AdS_4 and AdS_6 this partition function was independently computed also in [22, 23]

$$\mathcal{Z}_{\text{CHS},2}^+(\text{AdS}_4; q) = \frac{q^2(5+5q-4q^2)}{(1-q)^3}, \quad \mathcal{Z}_{\text{CHS},2}^+(\text{AdS}_6; q) = \frac{2q^3(7+7q+7q^2-3q^3)}{(1-q)^5}. \quad (3.18)$$

The CHS field in d dimensions is naturally associated to the massless higher spin (MHS) field in AdS_{d+1} with $\Delta_{d+1}^+ = d + s - 2$. It has the one-particle partition function [24–26]

$$\mathcal{Z}_{\text{MHS},s}^+(\text{AdS}_{d+1}; q) = \frac{g_s^{(d+1)} q^{d+s-2} - g_{s-1}^{(d+1)} q^{s+d-1}}{(1-q)^d}, \quad (3.19)$$

$$\tilde{\mathcal{Z}}_{\text{MHS},s}^-(\text{AdS}_{d+1}; q) = (-1)^d \mathcal{Z}_{\text{MHS},s}^+(\text{AdS}_{d+1}; q^{-1}), \quad (3.20)$$

where $g_s^{(d)}$ is given by (2.4). One can then check that

$$\tilde{\mathcal{Z}}_{\text{MHS},s}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{MHS},s}^+(\text{AdS}_{d+1}; q) = \tilde{\mathcal{Z}}_{\text{CHS},s}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_d; q), \quad (3.21)$$

which is another special case of (3.5). One can also verify the validity of (3.1) or, equivalently, (3.4) (see Appendix B).

3.3 Conformal symmetric tensors in $d = 4$

Next, let us discuss the conformal symmetric rank s tensor field (CST) in $d = 4$ considered in [27, 28]. This is a non-unitary theory that may be viewed as a maximal depth $r = s$ representative of the family of FT-type [29] conformal higher spin fields with rank $s - r$ tensor gauge invariance [30–32]. The CHS theory is the minimal depth case (i.e. case of maximal gauge invariance) when $r = 1$. The CST field has 2nd-derivative Lagrangian with scalar gauge invariance and corresponds to a "short" representation of $SO(2, 4)$ given by

$$\text{CST}_s = (1; \frac{s}{2}, \frac{s}{2}) - (1 - s; 0, 0). \quad (3.22)$$

The partition function for a CST field defined on AdS_4 is found to be [28] (cf. (2.1))

$$\mathcal{Z}_{\text{CST},s}(\text{AdS}_4) = \prod_{k=1}^s \left[\frac{\det \hat{\Delta}_0(2 - k - k^2)}{\det \hat{\Delta}_{k\perp}(2 + k)} \right]^{1/2}. \quad (3.23)$$

Using (2.2), (2.3) we then find for the one-particle partition function on thermal AdS_4

$$\mathcal{Z}_{\text{CST},s}^+(\text{AdS}_4; q) = -\tilde{\mathcal{Z}}_{\text{CST},s}^-(\text{AdS}_4; q^{-1}) = \frac{1}{(1-q)^3} \sum_{k=1}^s \left[(2k+1) q^2 - q^{k+2} \right]$$

$$= \frac{q^2 [s(s+2) - (s+1)^2 q + q^{s+1}]}{(1-q)^4} . \quad (3.24)$$

This 4d CST field corresponds to the maximal-depth partially massless (PM) totally symmetric spin s field in AdS_5 associated with the following combination of $SO(2,4)$ representations [31]⁸

$$\text{PM}_s^{(s)} = (3; \frac{s}{2}, \frac{s}{2}) - (3+s; 0, 0) , \quad (3.25)$$

for which (3.22) is a "shadow" counterpart. Then from (2.3) we get

$$\mathcal{Z}_{\text{PM}_s^{(s)}}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{PM}_s^{(s)}}^-(\text{AdS}_5; q^{-1}) = \frac{(s+1)^2 q^3 - q^{s+3}}{(1-q)^4} . \quad (3.26)$$

Comparing (3.24) and (3.26) we conclude that

$$\tilde{\mathcal{Z}}_{\text{PM}_s^{(s)}}^-(\text{AdS}_5; q) - \mathcal{Z}_{\text{PM}_s^{(s)}}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{CST},s}^-(\text{AdS}_4; q) + \mathcal{Z}_{\text{CST},s}^+(\text{AdS}_4; q) , \quad (3.27)$$

in agreement with (3.5). As in CHS case, one can also verify the validity of (3.1) also in the CST case. We shall further discuss the properties of 4d CST field partition functions in the next section.

4 From 5 to 4 to 3 dimensions

In this section we shall consider a special case of $d = 4$ where some relation simplify. We shall discuss further descent to 3 dimensions thus getting a "triple" of related fields: HS in 5, CF in 4, and $\widehat{\text{CF}}$ in 3 dimensions. In particular, starting with a massless higher spin field in AdS_5 one gets a conformal higher spin in 4d and then defining it on AdS_4 can further associate to it another conformal higher spin field in 3d. The latter turns out to have a non-local action describing zero number of dynamical degrees of freedom, i.e. giving trivial partition function.

4.1 $\text{AdS}_5 \rightarrow \text{AdS}_4$

Let us first consider the $d = 4$ version of the relation (3.1) between partition functions of some higher spin field in AdS_5 and the corresponding 4d conformal field defined on AdS_4 . Starting with a totally symmetric spin s HS field in AdS_5 corresponding to $SO(2,4)$ representation $(\Delta_5; \frac{s}{2}, \frac{s}{2})$ we may associate to it (in general, higher-derivative) conformal field in 4d that may also be represented (when defined on \mathbb{R}^4 or AdS_4) as a collection of 2nd-derivative fields with particular values of masses. Our proposal for such general relation is⁹

$$(\Delta_5; \frac{s}{2}, \frac{s}{2})_{\text{AdS}_5} \longrightarrow Z_{\text{CF},s}(\text{AdS}_4) = \prod_{s'=0}^s \prod_{k=0}^{\Delta_5-3} \left[\det \hat{\Delta}_{s' \perp} (M_{s',k}^2) \right]^{-1/2} , \quad (4.1)$$

⁸The subscript r in $\text{PM}_s^{(r)}$ denotes the depth. Here, we consider only the maximal case $r = s$.

⁹A similar "correspondence rule" in 6d was discussed in Appendix A of [33].

where $M_{s',k}^2 = s' - 2 - k(k+1)$ as in (3.15). Special HS fields with gauge invariance will require combinations of the above building blocks to take into account ghost field contributions. One can check that (4.1) is consistent with all special conformal fields in 4d that we have analysed directly: CHS, CST and also GJMS scalar fields (see Appendix C).

Given (4.1) one can then demonstrate the validity of the relation (3.1) (equivalent to (3.5) in the absence of gauge invariance) between the partition functions in AdS_5 and AdS_4 . For each factor in the r.h.s. of (4.1) we find from (2.2) that $\Delta_{4,k}^+ = k+2$ and thus applying (2.3) to both 5d and 4d cases we get

$$\mathcal{Z}_{(\Delta_5; \frac{s}{2}, \frac{s}{2})}^+(\text{AdS}_5; q) = (s+1)^2 \frac{q^{\Delta_5}}{(1-q)^4}, \quad (4.2)$$

$$\mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q) = \sum_{s'=0}^s \sum_{k=0}^{\Delta_5-3} (2s'+1) \frac{q^{k+2}}{(1-q)^3} = (s+1)^2 \frac{q^2 - q^{\Delta_5}}{(1-q)^4}. \quad (4.3)$$

Then using the expression (2.5) for $\tilde{\mathcal{Z}}^-$ we indeed verify (3.1), i.e.

$$\tilde{\mathcal{Z}}_{(\Delta_5; \frac{s}{2}, \frac{s}{2})}^-(\text{AdS}_5; q) - \mathcal{Z}_{(\Delta_5; \frac{s}{2}, \frac{s}{2})}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{CF}}^-(\text{AdS}_4; q) + \mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q). \quad (4.4)$$

To provide additional support for the correspondence rule (4.1) let us consider the CF partition function defined on S^4 instead of AdS_4 which may be viewed as a boundary of global AdS_5 . In that case we should get an analog of (2.7), i.e. the relation (1.2)

$$\log Z_{\text{HS}}^-(\text{AdS}_5) - \log Z_{\text{HS}}^+(\text{AdS}_5) = \log Z_{\text{CF}}(S^4). \quad (4.5)$$

One may check this relation by comparing the coefficient of the IR divergent term on the l.h.s. to the coefficient of the UV divergent term on the r.h.s., i.e. to the 4d conformal anomaly a-coefficient [2, 1]. According to Eq.(3.3) of [1], we get for the coefficient in the l.h.s.

$$a(\Delta_5; \frac{s}{2}, \frac{s}{2}) = -\frac{1}{720} (\Delta_5 - 2)^3 (s+1)^2 [3(\Delta_5 - 2)^2 - 5s^2 - 10s - 5]. \quad (4.6)$$

On the other hand, each $\det(-\nabla_s^2 + M^2)_{s\perp}$ in the product in (4.1) defined on S^4 gives the contribution (see Eq.(3.37) of [20])

$$a_{s\perp}(M^2) = \frac{1}{720} (2s+1) [30s^3 + 85s^2 + 10s - 58 - 30(s^2 - 2)M^2 - 15M^4]. \quad (4.7)$$

Then for the particular combination of the operators in (4.1) we get indeed

$$a(\Delta_5; \frac{s}{2}, \frac{s}{2}) = \sum_{k=0}^{\Delta_5-3} \sum_{s'=0}^s a_{s'\perp}(M_{s',k}^2). \quad (4.8)$$

Let us note that the relation (4.4), implied by the correspondence rule (4.1), should have a group theoretic interpretation. To see an indication of this, let us consider the "non-blind" characters χ_4 and χ_3 of massive representations of $SO(4,2)$ and $SO(3,2)$ respec-

tively [34]¹⁰

$$\begin{aligned}\chi_4(\Delta; j_1, j_2 | q, x, y) &= \frac{q^\Delta f_{\text{SU}(2)}(j_1 | x) f_{\text{SU}(2)}(j_2 | y)}{(1 - q x^{\frac{1}{2}} y^{\frac{1}{2}})(1 - q x^{\frac{1}{2}} y^{-\frac{1}{2}})(1 - q x^{-\frac{1}{2}} y^{\frac{1}{2}})(1 - q x^{-\frac{1}{2}} y^{-\frac{1}{2}})}, \\ \chi_3(\Delta; j | q, x) &= \frac{q^\Delta f_{\text{SU}(2)}(j | x)}{(1 - q)(1 - q x)(1 - q x^{-1})}, \quad f_{\text{SU}(2)}(j | x) \equiv \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}.\end{aligned}\quad (4.9)$$

Let us generalize (4.2) and define

$$\chi(\text{AdS}_5 | q, x) \equiv \chi_4(\Delta_5; \frac{s}{2}, \frac{s}{2} | q, x, x), \quad \chi(\text{AdS}_4 | q, x) \equiv \sum_{s'=0}^s \sum_{k=0}^{\Delta_5-3} \chi_3(k+2; s' | q, x). \quad (4.10)$$

Let us also denote by tilde the "charge conjugation", i.e. the replacement $q \rightarrow q^{-1}$, $x \rightarrow x^{-1}$. One can then check that

$$\tilde{\chi}(\text{AdS}_5 | q, x) - \chi(\text{AdS}_5 | q, x) = \tilde{\chi}(\text{AdS}_4 | q, x) + \chi(\text{AdS}_4 | q, x). \quad (4.11)$$

This reduces to (4.4) in the "blind" limit $x \rightarrow 1$. The fact that (4.11) holds also for generic argument x suggests that (4.4) has a group theoretic interpretation in terms of a map between representations of the corresponding 5d and 4d isometry groups.

4.2 Relation between partition functions on AdS_4 and $S^1 \times S^3$

As already mentioned above, given a conformal field in AdS_d we may make a Weyl transformation to replace AdS_d by half of the Einstein Universe $R \times S^{d-1}$ and then represent the partition function in $R \times S^{d-1}$ in terms of the partition function in AdS_d with two possible choices of boundary conditions. In the case of thermal quotients that leads to the relation (3.2).

In the special case of AdS_4 it was observed in [4] that the two choices (+ and -) of the possible boundary conditions are equivalent, i.e. the corresponding higher spin representations are equivalent for $s > 0$, with spin 0 (scalar) case being an exception.¹¹ This suggests that for any conformal field not containing a scalar component we should have the equality between partition functions corresponding to the two alternative boundary conditions

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_4; q) = \mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q), \quad (4.12)$$

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_4; q) = \tilde{\mathcal{Z}}_{\text{CF}}^-(\text{AdS}_4; q) + \sigma_4(q), \quad \tilde{\mathcal{Z}}_{\text{CF}}^-(\text{AdS}_4; q) = -\mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q^{-1}) \quad (4.13)$$

Then the relation (3.2) should simplify in $d = 4$ case to

$$\mathcal{Z}_{\text{CF}}(S^1 \times S^3; q) = 2 \mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q). \quad (4.14)$$

¹⁰Here x and y are chemical potentials for charges corresponding to other Cartan generators in addition to the dilatation operator.

¹¹More generally, the fact that highest weight unitary representation of $\mathfrak{so}(d, 2)$ algebra that admits extension to $\mathfrak{so}(d+1, 2)$ conformal algebra has two inequivalent extensions was demonstrated in [35]; for scalar field these representations are not equivalent as representations of $\mathfrak{so}(d, 2)$ while for spin $s > 0$ fields they are.

This identity can be verified directly for the CHS or CST fields as follows.

The CHS partition function on AdS_4 is a special case of (3.14)¹²

$$Z_{\text{CHS},s}(\text{AdS}_4) = \prod_{k=0}^{s-1} \left[\frac{\det \hat{\Delta}_{k\perp}(k - (s-1)(s+2))}{\det \hat{\Delta}_{s\perp}(s - (k-1)(k+2))} \right]^{1/2}. \quad (4.15)$$

From (2.2) we see that the fields corresponding to terms in the numerator have $\Delta_k^+ = s+2$, while the terms in the denominator give $\Delta_k^+ = k+2$. The corresponding one-particle partition function is then a $d=4$ case of (3.17)

$$\mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_4; q) = \sum_{k=0}^{s-1} \frac{(2s+1)q^{k+2} - (2k+1)q^{s+2}}{(1-q)^3} = \frac{(2s+1)q^2 - (s+1)^2 q^{s+2} + s^2 q^{s+3}}{(1-q)^4}. \quad (4.16)$$

Comparing this to the CHS partition function on $S^1 \times S^3$ given in Eq. (4.8) of [1] we find that indeed

$$\mathcal{Z}_{\text{CHS},s}(S^1 \times S^3; q) = 2 \mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_4; q). \quad (4.17)$$

In the case of the CST field the partition function on $S^1 \times S^3$ was found in [28]. Comparing to (3.24) we conclude again that

$$\mathcal{Z}_{\text{CST},s}(S^1 \times S^3; q) = 2 \mathcal{Z}_{\text{CST},s}^+(\text{AdS}_4; q). \quad (4.18)$$

The 4d relation (4.14) may be extended also to the fermionic CHS fields, see Appendix (D).

This relation (4.14) is not, however, true for a conformal scalar and thus also for any conformal theory with AdS_4 partition function containing a conformal scalar factor. In particular, it is not true for the 4-derivative conformal scalar field as follows from the comparison of (C.12) and (C.8) in Appendix C. Another counter-example is the conformal theory of an antisymmetric rank 2 tensor discussed in Appendix E.¹³

4.3 Further descent: $\text{AdS}_4 \rightarrow \mathbb{R} \times S^2$

Given a conformal field in 4d related to some higher spin field in AdS_5 we may define it on AdS_4 and then further associate to it another conformal field $\widehat{\text{CF}}$ at the AdS_4 boundary $\mathbb{R} \times S^2$. This gives a triplet of fields

$$\text{HS on AdS}_5 \longrightarrow \text{CF on AdS}_4 \longrightarrow \widehat{\text{CF}} \text{ on } \mathbb{R} \times S^2.$$

Then the thermal partition functions of CF and $\widehat{\text{CF}}$ are related by (1.3) or (2.7), i.e. for the second step we get

$$\mathcal{Z}_{\text{CF}}^-(\text{AdS}_4; q) - \mathcal{Z}_{\text{CF}}^+(\text{AdS}_4; q) = \mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^2; q). \quad (4.19)$$

Combining this with (4.12) we conclude that $\widehat{\text{CF}}$ should have zero partition function,

$$\mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^2; q) = 0. \quad (4.20)$$

¹²While here we have a scalar field contribution at $k=0$ this is not a conformal massless scalar but a ghost field needed to guarantee the conformal invariance of the spin s CHS field.

¹³Here the presence of a scalar components is apparent also in the approach developed in [36].

Then the total partition function of the theory defined on "boundary of the boundary" is $Z = 1$, i.e. the resulting 3d conformal theory $\widehat{\text{CF}}$ should be "trivial" or "topological". We will also define $\widehat{\text{CF}}$ on AdS_3 and then find, in agreement with (3.3), that

$$\mathcal{Z}_{\widehat{\text{CF}}}^+(\text{AdS}_3; q) = \mathcal{Z}_{\widehat{\text{CF}}}^-(\text{AdS}_3; q) = 0. \quad (4.21)$$

Before giving some explicit examples let us first recall that a totally symmetric spin s CHS field ϕ_s in d dimensions has the action

$$S_{\text{CHS}_s} = \int d^d x \phi_s P_s \partial^{2s+d-4} \phi_s = \int d^d x C_s \partial^{d-4} C_s, \quad (4.22)$$

where P_s is a projector onto transverse traceless tensors and $C_s \sim \partial^s \phi_s$ is gauge-invariant field strength (generalized Maxwell or Weyl tensor). The number of the corresponding dynamical degrees of freedom is [16, 21]

$$\nu_{s,d} = \frac{(d-3)(2s+d-2)(2s+d-4)(s+d-4)!}{2(d-2)!s!}. \quad (4.23)$$

Eq. (4.22) is local for even $d \geq 4$ (where $\partial^2 = \square$ enters in positive power) but can be formally defined also for odd d . The case of $d = 3$ is special in that the number of dynamical degrees of freedom (4.23) vanishes, while (4.22) takes a non-local form

$$S_{\text{CHS}_s} = \int d^3 x \phi_s P_s \square^{s-1/2} \phi_s = \int d^3 x C_s \square^{-1/2} C_s. \quad (4.24)$$

Let us also recall that the CHS action in d dimensions may be viewed as an induced one [37] from a free CFT_d : if ϕ_s is coupled to a spin- s conserved current J_s then the kinetic term of ϕ_s is determined by the 2-point function $\langle J_s(x) J_s(x') \rangle$.¹⁴ Insisting on locality one may consider a Chern-Simons type action for the corresponding 3d CHS field that may be induced from chiral 3d fermions (see [40–42] and [43–45] for $s = 2$). For a more natural parity-even case induced from a free 3d scalar CFT we get in momentum space $\langle J_s J_s \rangle = \frac{k_s}{\sqrt{p^2}} \tilde{P}_s(p)$, where $\tilde{P}_s(p)$ is Fourier transform of the transverse traceless projector in (4.22) (i.e. a symmetrized and traceless product of s factors of $(\tilde{P}_1)_\mu^\nu = \delta_\mu^\nu - \frac{p_\mu p^\nu}{p^2}$). The corresponding parity even 3d CHS action is then given by (4.22).

4.3.1 Spin 1

Let us now illustrate (4.20) turn to some special cases and start with a massless spin 1 gauge field in AdS_5 that has $\Delta_5^+ = 2 + s = 3$ and is associated with the following combination of $SO(2, 4)$ representations $\text{MHS}_1(\text{AdS}_5) = (3; \frac{1}{2}, \frac{1}{2}) - (4; 0, 0)$. The corresponding 4d boundary field is the $s = 1$ CHS field, i.e. the standard Maxwell theory (cf. (4.22)). Its partition function when defined on AdS_4 is a special case of (3.14)

$$Z_{\text{CHS},1}(\text{AdS}_4) = \left[\frac{\det \hat{\Delta}_0(0)}{\det \hat{\Delta}_{1\perp}(3)} \right]^{1/2}. \quad (4.25)$$

¹⁴In general, in 3d there are two possible conformally invariant tensor structures that may appear in a two-point function of a conserved current J_s : a non-local parity-even and a local parity-odd one (see, e.g., [38, 39, 2]).

Here each operator is in turn associated with a conformal field at the $R \times S^2$ boundary (we get a scalar with $\Delta_4^+ = 3$ and a transverse vector with $\Delta_4^+ = 2$). The corresponding combination of $SO(2,3)$ representations $(\Delta_4^+; j)$ is¹⁵

$$\text{CHS}_1(\text{AdS}_4) = (2; 1) - (3; 0) = \text{MHS}_1(\text{AdS}_4) . \quad (4.26)$$

Thus the field at the 3d boundary should be the $s = 1$ member of the 3d CHS family (4.24) with a non-local action (cf. [46, 2])

$$S_{\text{CHS}_1} = \int d^3x F_{\mu\nu} \square^{-1/2} F_{\mu\nu} . \quad (4.27)$$

This theory is effectively topological, having no dynamical degrees of freedom (in agreement with (4.23)). One can see this explicitly, e.g., by computing the corresponding partition function in flat 3d space¹⁶

$$Z_{\text{CHS}_1}(\mathbb{R}^3) = \left[\frac{\det \square}{\det(\partial \square^{-1/2} \partial)_{1\perp}} \right]^{1/2} = \left[\frac{\det \square}{\det(\square^{1/2})_{1\perp}} \right]^{1/2} \stackrel{3d}{=} \left[\frac{\det \square}{\det(\square^{1/2})^2} \right]^{1/2} = 1 . \quad (4.28)$$

As there is no conformal anomaly in odd dimensions the same should be true also for all conformally flat spaces, e.g., AdS_3

$$Z_{\text{CHS}_1}(\text{AdS}_3) = 1 . \quad (4.29)$$

Defining the 4d Maxwell field on AdS_4 we get from (A.6)

$$\begin{aligned} \tilde{\mathcal{Z}}_{\text{CHS}_1}^-(\text{AdS}_4; q) - \mathcal{Z}_{\text{CHS}_1}^+(\text{AdS}_4; q) &= -\mathcal{Z}_{\text{CHS}_1}^+(\text{AdS}_4; q^{-1}) - \mathcal{Z}_{\text{CHS}_1}^+(\text{AdS}_4; q) \\ &= -\frac{3/q^2 - 1/q^3}{(1-1/q)^3} - \frac{3q^2 - q^3}{(1-q)^3} = -1 . \end{aligned} \quad (4.30)$$

This -1 is precisely what is removed by the $\sigma_{\text{CHS}_{1,4}}(q)$ term in (2.6) in agreement with (4.12) so that we get $\mathcal{Z}_{\text{CHS}_1}(S^1 \times S^2; q) = 0$ as a special case of (4.20).¹⁷

We can also explicitly check the equality (3.4) of the σ -terms. For the $\text{MHS}_s(\text{AdS}_5)$ theory the $\sigma_{\text{MHS}_{s,5}}(q)$ term for a general spin s may be found in Eq. (5.5) of [1] and for $s = 1$ it is equal to 1, i.e. is indeed the same as the above $\sigma_{\text{CHS}_{1,4}}(q)$.

¹⁵Here j is the $SO(3)$ angular momentum. In general, $\text{MHS}_s(\text{AdS}_4) = (1 + s; s) - (2 + s; s - 1)$, i.e. it corresponds to a spin s field with gauge invariance with spin $s - 1$ parameter. The partition function for a massive $SO(2,3)$ representation $(\Delta_4; s)$ is given by $\mathcal{Z}_{(\Delta_4; s)}^+(q) = (2s + 1) \frac{q^{\Delta_4}}{(1-q)^3}$.

¹⁶Here the measure contribution from the decomposition $A_\mu = A_{\mu\perp} + \partial_\mu \phi$ cancels against the kinetic operator contribution. Note also that in 3d one can dualize $F_{\mu\nu} \frac{1}{\square^{1/2}} F_{\mu\nu}$ to a scalar with kinetic term $\phi \square^{3/2} \phi$ but the corresponding partition function is still 1 as the scalar determinant is cancelled by the measure contribution coping from integrating out the auxiliary field $F_{\mu\nu}$.

¹⁷Indeed, the partition function on $S^1 \times S^{d-1}$ for the CHS_1 Maxwell field with the action in (4.22), i.e. $\int d^d x F^{\mu\nu} \square^{\frac{d-4}{2}} F_{\mu\nu}$, was already computed in [1] with the general expression being

$$\mathcal{Z}_{\text{CHS}_1}(S^1 \times S^{d-1}; q) = 1 - \frac{1-dq + dq^{d-1} - q^d}{(1-q)^d} .$$

This vanishes for $d = 3$.

4.3.2 Spin 2

Let us now consider the $s = 2$ case, i.e. start with the MHS theory in AdS_5 describing massless rank-2 tensor with $\Delta_5^+ = 2 + s = 4$ and spin 1 gauge invariance parameter, i.e. associated with the following combination of $SO(2, 4)$ representations $\text{MHS}_2(\text{AdS}_5) = (4; 1, 1) - (5; \frac{1}{2}, \frac{1}{2})$. The dual conformal field in 4d is the $s = 2$ CHS theory, i.e. Weyl gravity. Its partition function on AdS_4 is a special case of (3.23), i.e. [47, 48, 29]

$$Z_{\text{CHS}_2}(\text{AdS}_4) = \left[\frac{\det \widehat{\Delta}_0(-4) \det \widehat{\Delta}_{1\perp}(-3)}{\det \widehat{\Delta}_{2\perp}(4) \det \widehat{\Delta}_{2\perp}(2)} \right]^{1/2}. \quad (4.31)$$

Using (2.2) the values of the scaling dimensions Δ_4^+ corresponding to each factor in (4.31) are (cf. (2.1))

	$\widehat{\Delta}_{2\perp}(4)$	$\widehat{\Delta}_{2\perp}(2)$	$\widehat{\Delta}_{1\perp}(-3)$	$\widehat{\Delta}_0(-4)$
Δ_4^+	2	3	4	4

(4.32)

This means that the equivalent combination of $SO(2, 3)$ representations is¹⁸

$$\text{CHS}_2(\text{AdS}_4) = \text{MHS}_2(\text{AdS}_4) \oplus \text{PM}_2^{(2)}(\text{AdS}_4) = [(3; 2) - (4; 1)] \oplus [(2; 2) - (4; 0)]. \quad (4.33)$$

Indeed, the Weyl graviton on AdS_4 is a combination of Einstein graviton and a partially massless spin 2 field with scalar gauge invariance [49, 50, 20].

The 3d conformal theory "induced" by 4d Weyl graviton at the boundary of AdS_4 thus contains two parts. From the Einstein graviton $\text{MHS}_2(\text{AdS}_4)$ we get a conformally invariant 3d CHS₂ or Weyl theory with parity-even non-local linearized action (4.24), i.e. $\int d^3x C_2 \square^{-1/2} C_2$ (see also [39]).¹⁹ From the partially massless field $\text{PM}_2^{(2)}(\text{AdS}_4)$ we get CST₂ field representing a non-unitary 3d symmetric tensor $\varphi_{\mu\nu}$ with scalar gauge invariance $\delta\varphi_{\mu\nu} = \partial_\mu \partial_\nu \epsilon$. This theory has a non-local action $\int d^3x \varphi_2 P_2 \square^{1/2} \varphi_2$ where P_2 is an appropriate projector ensuring scalar gauge invariance.²⁰ In summary, the conformal field combination corresponding to (4.33) on flat 3d boundary is²¹

$$\widehat{\text{CF}}(\mathbb{R}^3) = \text{CHS}_2(\mathbb{R}^3) \oplus \text{CST}_2(\mathbb{R}^3). \quad (4.34)$$

Let us now show that this system has zero total number ν of dynamical degrees of freedom. Indeed, the CHS₂ field has $\nu = 0$ according to (4.23). For CST₂ the flat space partition function is

$$Z_{\text{CST}_2}(\mathbb{R}^3) = \left[\frac{\det \square_{1\perp} (\det \square_0)^2}{(\det \square^{1/2})_{2\perp} (\det \partial \square^{1/2} \partial)_{1\perp}} \right]^{1/2} = \left[\frac{\det \square_{1\perp} (\det \square_0)^2}{(\det \square_{2\perp})^{1/2} (\det \square_{1\perp})^{3/2}} \right]^{1/2} = 1. \quad (4.35)$$

¹⁸In general, for a partially massless spin s field we have $\text{PM}_s^{(s)}(\text{AdS}_4) = (2; s) - (2 + s; 0)$.

¹⁹The full non-linear action of Weyl-invariant gravity in 3d with quadratic part given by $\int d^3x C_2 \square^{-1/2} C_2$ where C_2 is linearized Weyl tensor can be obtained as an induced one corresponding to a conformally coupled scalar in 3d, i.e. as $\log \det(-\nabla^2 + \frac{1}{8}R)$. Since there is no Weyl anomaly in 3 dimensions this non-local functional of the metric will be both reparametrization and Weyl invariant.

²⁰Indeed, from (4.33) we find that the canonical dimension of $\varphi_{\mu\nu}$ is $d - \Delta^+ = 3 - 2 = 1$.

²¹An alternative way of obtaining (quadratic) action for this set of 3d fields is to start with the (linearized) Weyl gravity in AdS_4 space, specify separate boundary conditions for the 4d graviton and partially massless mode (in terms of 3d graviton and 3d CST₂ field respectively) and then evaluate the 4d action on the solution of the equations of motion. This procedure may have non-linear generalization if one starts with the full non-linear 4d Weyl gravity action and considers a generic asymptotically AdS_4 background.

Here the numerator is the Jacobian for the change of variables $\varphi_{\mu\nu} = \varphi_{\mu\nu}^\perp + \partial_{(\mu} V_{\nu)}^\perp + (\partial_\mu \partial_\nu - \frac{1}{3} \delta_{\mu\nu} \partial^2) \gamma$. The denominator is from the action $\int d^3x \varphi_2 P_2 \square^{1/2} \varphi_2$ where the scalar component γ drops out due to gauge invariance. Thus $\nu(\text{CST}_2)|_{d=3} = \frac{3}{2} \times 2 + \frac{1}{2} \times 2 - (2+2) = 0$. We have used that in three dimensions a contribution $(\det \square_{s\perp})^{-1/2}$ in the partition function is equivalent²² to $(\det \square_0)^{-1}$ (i.e. $\nu = 2$) for $s > 0$.

Let us note for completeness that (4.35) may be generalized to any spin $s > 0$ as follows

$$Z_{\text{CST}_s}(\mathbb{R}^3) = \left[\frac{\det \square_{s-1,\perp} \det \square_{s-2,\perp}^2 \cdots \det \square_s^s}{\det \square_{s\perp}^{1/2} \det \square_{s-1,\perp}^{3/2} \cdots \det \square_{1\perp}^{s-1/2}} \right]^{1/2} = 1. \quad (4.36)$$

Thus as for a CHS field, the total number of d.o.f. of a 3d CST field is zero for any s : $\nu = 2 \sum_{n=1}^s (n - \frac{1}{2}) - (2 \sum_{n=1}^{s-1} n + s) = 0$.

At the level of partition function, the decomposition (4.33) implies

$$\mathcal{Z}_{\text{CHS}_2}^+(\text{AdS}_4; q) = \frac{5q^3 - 3q^4}{(1-q)^3} + \frac{5q^2 - q^4}{(1-q)^3} = \frac{q^2(5+5q-4q^2)}{(1-q)^3}. \quad (4.37)$$

We may compute the partition function of $\widehat{\text{CF}} = \text{CHS}_2 \oplus \text{CST}_2$ as in (2.7),(2.6)

$$\begin{aligned} \mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^2; q) &= -\mathcal{Z}_{\text{CHS}_2}^+(\text{AdS}_4; q^{-1}) - \mathcal{Z}_{\text{CHS}_2}^+(\text{AdS}_4; q) + \sigma_{\text{CHS}_2,4}(q) \\ &= -4(q + q^{-1}) - 7 + \sigma_{\text{CHS}_2,4}(q) = 0, \end{aligned} \quad (4.38)$$

which is in agreement with (4.20) (see also Appendix B). We used that, as one can check, $\sigma_{\text{CHS}_2,4}(q) = 4(q + q^{-1}) + 7$.²³ Again, we can compare the expression for $\sigma_{\text{CHS}_2,4}(q)$ with the $\sigma_{\text{MHS}_{2,5}}(q)$ term in Eq. (5.5) of [1] for $s = 2$ and thus verify the relation (3.4).

5 Concluding remarks

The "(AdS/CFT)² = 0" relation (4.20) is special to $d = 4$ case because we used (4.12) to obtain it. In $d > 4$ case we expect to find a more complicated picture. For example, suppose we start with a massless HS field in AdS_7 . Then at the boundary we get conformal HS field in 6d and can define it on AdS_6 thus associating to it some other conformal field CF_5 at the boundary of AdS_6 . We can then continue the descent, i.e. define CF_5 on AdS_5 and associate to it another conformal field CF_4 at the boundary of AdS_5 , etc. One may then look for some new identities between partition functions of these fields in addition to (2.7) and (3.1). For example, one can check that in the spin 1 case CF_5 appears to be represented by a combination of 5d CHS_1 field (with non-local action $\int d^5x F_{\mu\nu} \square^{1/2} F_{\mu\nu}$ as in (4.22)) and an extra field, such that one ends up with CF_4 being just the standard CHS_1 Maxwell field. Details of the corresponding relations between partition functions remain to be studied.

²²The number of components of a totally symmetric traceless rank- s tensor ϕ_s in d dimensions is $N_s = \binom{s+d-1}{s} - \binom{s+d-3}{s-2}$, i.e. $N_s|_{d=3} = 2s+1$. The number of components of transverse ($\partial \cdot \phi_{s\perp} = 0$) traceless rank $s > 0$ tensor is $N_{s,\perp} = N_s - N_{s-1} \xrightarrow{3d} 2$.

²³Here we can use the expression for $\mathcal{Z}_{\text{CHS}_2}(S^1 \times S^{d-1}; q)$ in Eq. (5.17) of [1] and check that it vanishes for $d = 3$.

Given the general relations (1.3) and (1.8) between partition functions of particular higher spin fields one may apply them to theories containing infinite number of spins. For example, the Vasiliev-type theory in AdS_{d+1} (containing a scalar and totally symmetric MHS spin $1, 2, \dots$ fields and dual to the singlet sector of free $U(N)$ scalar theory in 4d) is naturally associated to the CHS theory of all conformal spins $s = 0, 1, 2, \dots$ in d dimensions (with linearized action (4.22)). Summing over all spins the relation (1.8) should trivialise. Indeed, for the MHS theory we find [26] (spin 0 field here has $\Delta^+ = d - 2$)

$$\mathcal{Z}_{\text{MHS}}^+(\text{AdS}_{d+1}; q) = \sum_{s=0}^{\infty} \mathcal{Z}_{\text{MHS},s}^+(\text{AdS}_{d+1}; q) = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}, \quad (5.1)$$

$$\tilde{\mathcal{Z}}_{\text{MHS}}^-(\text{AdS}_{d+1}; q) = (-1)^d \mathcal{Z}_{\text{MHS}}^+(\text{AdS}_{d+1}; q^{-1}) = (-1)^d \mathcal{Z}_{\text{MHS}}^+(\text{AdS}_{d+1}; q). \quad (5.2)$$

Thus, e.g., for $d = 4$ the l.h.s. of (1.8) vanishes after summing over all spins (the σ term in \mathcal{Z}^- in (1.5) drops out being symmetric under $q \rightarrow q^{-1}$). At the same time, the summed CHS partition function on thermal AdS_d in (3.17) appears to be divergent (cf. [1]). In four dimensions $\mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_4; q)$ is given by (4.16) and the divergence is due to the term $\sim (2s+1)q^2$ in the numerator that is not suppressed at large s . Nevertheless, applying the analog of the standard ζ -function regularization (i.e. $\sum_{s=1}^{\infty} 1 = \zeta(0) = -\frac{1}{2}$, etc.), we obtain for the regularized expression of the sum over spins

$$\mathcal{Z}_{\text{CHS}}^+(\text{AdS}_4; q) = \lim_{z \rightarrow 0} \sum_{s=1}^{\infty} s^z \mathcal{Z}_{\text{CHS},s}^+(\text{AdS}_4; q) = -\frac{2}{3} \frac{q^2(q^2 + 4q + 1)}{(1-q)^6}, \quad (5.3)$$

$$\mathcal{Z}_{\text{CHS}}^+(\text{AdS}_4; q) = \mathcal{Z}_{\text{CHS}}^+(\text{AdS}_4; q^{-1}), \quad \tilde{\mathcal{Z}}_{\text{CHS}}^-(\text{AdS}_4; q) = -\mathcal{Z}_{\text{CHS}}^+(\text{AdS}_4; q). \quad (5.4)$$

Thus the r.h.s. of (1.8) also vanishes. The same conclusion is reached also in general even dimension $d > 4$ once the non-trivial spin 0 contribution is included in the sum in (5.3).

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A Reconstructing $\mathcal{Z}_{\text{CF}}^+(\text{AdS}_d)$ from $\mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1})$

Here we shall reverse the logic: assume that the relation (3.5) is true and use it to determine the partition function $\mathcal{Z}_{\text{CF}}^+(\text{AdS}_d)$ from the knowledge of $\mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1})$ just by doing algebraic manipulations.

These partition functions have the following general form

$$\mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) = \frac{P(q) q^{\frac{d}{2}}}{(1-q)^d}, \quad \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) = \frac{F(q) q^{\frac{d-1}{2}}}{(1-q)^{d-1}}, \quad (\text{A.1})$$

where $P(q)$ and $F(q)$ are finite sums of non-negative powers of q (in AdS_d we have $\Delta_d^+ \geq \frac{1}{2}(d-1)$, cf. (2.3)). Eq. (3.5) or (3.6) implies that

$$F(q) + F(q^{-1}) = \frac{\sqrt{q}}{1-q} [P(q^{-1}) - P(q)]. \quad (\text{A.2})$$

The r.h.s. of (A.2) may be expanded in a Laurent series around $q = 0$ and comparing with the l.h.s., we may then determine $F(q)$.

Let us consider some examples. Let us start with the conformal scalar in AdS_4 which corresponds to a massive scalar in AdS_5 with $\Delta_5^+ = 3$, i.e. (cf. (3.7))

$$\mathcal{Z}_0^+(\text{AdS}_5; q) = \frac{q^3}{(1-q)^4}, \quad P(q) = q. \quad (\text{A.3})$$

Then (A.2) gives

$$F(q) + F(q^{-1}) = \frac{q+1}{\sqrt{q}} = q^{-1/2} + q^{1/2} \longrightarrow F(q) = q^{1/2}, \quad (\text{A.4})$$

and therefore, in agreement with (3.10), $\mathcal{Z}_{\text{CS}}^+(\text{AdS}_4; q) = \frac{q^2}{(1-q)^3}$.

Another example is the spin 1 field in AdS_5 corresponding to spin 1 CHS field (i.e. Maxwell field) in AdS_4 . From (3.19) we have

$$\mathcal{Z}_{\text{MHS}_1}^+(\text{AdS}_5; q) = \frac{4q^3 - q^4}{(1-q)^4}, \quad P(q) = 4q - q^2. \quad (\text{A.5})$$

Then, (A.2) gives

$$F(q) + F(q^{-1}) = -q^{3/2} - q^{-3/2} + 3q^{1/2} + 3q^{-1/2} \longrightarrow F(q) = 3q^{1/2} - q^{3/2}, \quad (\text{A.6})$$

and therefore we get, in agreement with (3.17), $\mathcal{Z}_{\text{CHS}_1}^+(\text{AdS}_4; q) = \frac{3q^2 - q^3}{(1-q)^3}$.

Our third example is a non-unitary CFT represented by a vector V_μ in 6d with 2nd-derivative kinetic term. This is a special $s = 1$ case of CST family of conformal fields described by rank- s symmetric tensors $\varphi_{\mu_1 \dots \mu_s}$ which in $d = 6$ have no gauge invariance. As discussed in [33], this CF is induced by a higher spin field in AdS_7 transforming in the $(\Delta; h_1, h_2, h_3) = (4; 1, 0, 0)$ representation of $SO(2, 6)$. Taking into account that $\dim[1, 0, 0] = 6$, we get (cf. (2.3))

$$\mathcal{Z}_{\text{HS}}^+(\text{AdS}_7; q) = \frac{6q^4}{(1-q)^6}, \quad P(q) = q. \quad (\text{A.7})$$

Then from (A.2) we have again $F(q) = q^{1/2}$ as in (A.4) and thus we predict that

$$\mathcal{Z}_V^+(\text{AdS}_6; q) = \frac{6q^3}{(1-q)^5}. \quad (\text{A.8})$$

This expression follows indeed from the general expression for the V_μ partition function on S^6 or on AdS_6 given in Eq. (A.4) of [33]

$$Z_V(\text{AdS}_6) = \left[\det \hat{\Delta}_{1\perp}(7) \det \hat{\Delta}_0(6) \right]^{-1/2}. \quad (\text{A.9})$$

Applying (2.2) to the operators here we find that we have the same $\Delta_6^+ = 3$ and $\Delta_6^- = 2$ for both factors. However, the number of degrees of freedom of a transverse vector in 6d is $6 - 1 = 5$ while the scalar contributes only one. Thus the numerator of $\mathcal{Z}_V^+(\text{AdS}_6; q)$ should be $(5 + 1)q^3 = 6q^3$, in agreement with (A.8).

B σ -term relation in Eq. (3.4)

Let us now use the general structure (A.1),(A.2) of the higher spin partition functions in AdS_{d+1} and the corresponding conformal field partition functions in AdS_d to justify the equality in (3.4).

According to (2.6),(2.7) we have for the partition function of CF on $S^1 \times S^d$ and the partition of another conformal field $\widehat{\text{CF}}$ (the one associated to CF on AdS_d) on $S^1 \times S^{d-1}$

$$\mathcal{Z}_{\text{CF}}(S^1 \times S^d; q) = \tilde{\mathcal{Z}}_{\text{HS}}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_{d+1}; q) + \sigma_{\text{HS}, d+1}(q), \quad (\text{B.1})$$

$$\mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^{d-1}; q) = \tilde{\mathcal{Z}}_{\text{CF}}^-(\text{AdS}_d; q) - \mathcal{Z}_{\text{CF}}^+(\text{AdS}_d; q) + \sigma_{\text{CF}, d}(q). \quad (\text{B.2})$$

Using $\tilde{\mathcal{Z}}^-(\text{AdS}_n; q) = (-1)^{n+1} \mathcal{Z}^+(\text{AdS}_n; q^{-1})$, and also (A.2), we find

$$\mathcal{Z}_{\text{CF}}(S^1 \times S^d; q) = \frac{q^{\frac{d-1}{2}} [F(q^{-1}) + F(q)]}{(1-q)^{d-1}} + \sigma_{\text{HS}, d+1}(q), \quad (\text{B.3})$$

$$\mathcal{Z}_{\widehat{\text{CF}}}(S^1 \times S^{d-1}; q) = \frac{q^{\frac{d-1}{2}} [F(q^{-1}) - F(q)]}{(1-q)^{d-1}} + \sigma_{\text{CF}, d}(q). \quad (\text{B.4})$$

The role of the σ -terms is to remove the negative powers of q in the expansion around $q = 0$ of the r.h.s. of (B.3),(B.4) as such terms cannot be present on the l.h.s. that can be computed using operator counting method and thus should have only positive powers of q . Such terms come only from the $F(q^{-1})$ term that is the same in the two lines of (B.3). This then implies that

$$\sigma_{\text{HS}, d+1}(q) = \sigma_{\text{CF}, d}(q). \quad (\text{B.5})$$

To give an example, let us consider the CHS_2 field in 4d for which from (4.37) we have

$$\mathcal{Z}_{\text{CHS}_2}^+(\text{AdS}_4; q) = \frac{q^2(5+5q-4q^2)}{(1-q)^3} \longrightarrow F(q) = 5q^{1/2} + 5q^{3/2} - 4q^{5/2}. \quad (\text{B.6})$$

Then (B.3) contains

$$\frac{q^{3/2} F(q^{-1})}{(1-q)^3} = \frac{-4+5q+5q^2}{q(1-q)^3} = -4q^{-1} - 7 - 4q + 5q^2 + \dots, \quad (\text{B.7})$$

leading to the same result as in [1]

$$\sigma_{\text{MHS}_2, 5}(q) = 4q^{-1} + 7 + 4q, \quad (\text{B.8})$$

with the same expression also for $\sigma_{\text{CHS}_2, 4}(q)$.

C Higher derivative conformal scalar fields

Here we will illustrate the relations (3.1),(3.2),(3.5) on the example of Weyl-covariant scalar theory with kinetic operator $\widehat{\Delta}_{(2r)} = -(\nabla^2)^r + \dots$ where dots stand for curvature dependent terms.

The GJMS operators $\widehat{\Delta}_{(2r)}$ naturally exist in a technical sense for $r \leq d/2$ (see [51] and refs. there). This means that their definition in generic dimension d involves terms

whose coefficients have poles in d when $r > d/2$. For instance, for $r = 3$, there are tensor structures with coefficient $\frac{1}{d-4}$ forbidding a naive extension to 4d case. These obstructions are proportional to Bach tensor and vanish for the Einstein spaces with $R_{mn} = \frac{R}{d} g_{mn}$. In this case it is possible to construct generalized Gover-GJMS operators (defined beyond critical order) but the resulting expressions are non-natural in technical sense [52]. For all orders (below, at, and beyond critical order) the Gover-GJMS operators factorise as in (C.10) below. In the conformally flat spaces (that need not be Einstein in general), there is no obstruction in going beyond the critical order $r = d/2$.

C.1 Partition function on $S^1 \times S^3$

Let us first consider the general case of the space $S^q \times S^p$ is conformally flat if defined with indefinite (p, q) signature metric (here the spheres have unit radius and $d = p + q$). Then one can show that for $r = 2N$ [53]

$$\widehat{\Delta}_{(4N)} = \prod_{k=1}^N \left[(\mathcal{O}_p - \mathcal{O}_q)^2 - 2(2k-1)^2(\mathcal{O}_p + \mathcal{O}_q) + (2k-1)^4 \right], \quad (\text{C.1})$$

$$\widehat{\Delta}_{(4N+2)} = (\mathcal{O}_p - \mathcal{O}_q) \prod_{k=1}^N \left[(\mathcal{O}_p - \mathcal{O}_q)^2 - 2(2k)^2(\mathcal{O}_p + \mathcal{O}_q) + (2k)^4 \right] \quad (\text{C.2})$$

where $\mathcal{O}_p \equiv -\nabla_{S^p}^2 + \frac{1}{4}(p-1)^2$. For example, in the special case of $\Delta_{(4)}$ in $d = 4$ we have for a general curved background [49, 54]

$$\widehat{\Delta}_{(4)} = -(\nabla^2)^2 + 2(R^{mn} - \frac{1}{3}g^{mn}R)\nabla_m\nabla_n. \quad (\text{C.3})$$

Then for $S^2 \times S'^2$ with $(++--)$ signature we get²⁴

$$\widehat{\Delta}_{(4)} = (-\nabla_{S^2}^2 + \nabla_{S'^2}^2)^2 + 2(\nabla_{S^2}^2 + \nabla_{S'^2}^2) = (\mathcal{O}_2 - \mathcal{O}_2')^2 - 2(\mathcal{O}_2 + \mathcal{O}_2') + 1, \quad (\text{C.4})$$

in agreement with (C.1) where $\mathcal{O}_2 = -\nabla_{S^2}^2 + \frac{1}{4}$.

For conformally flat but not Einstein space $S^1 \times S^3$ with $(-+++)$ signature we have for the spectra of \mathcal{O}_p in (C.1)

$$\mathcal{O}_1 \rightarrow w, \quad \mathcal{O}_3 \rightarrow n(n+2) + 1 = (n+1)^2, \quad n = 0, 1, 2, \dots \quad (\text{C.5})$$

For $r = 2N$ the factorisation of $\Delta_{(4N)}$ in (C.1) leads to the energy eigenvalues w represented by

$$w_n = n + 2 - r + 2k, \quad k = 0, \dots, r-1, \quad (\text{C.6})$$

so that the final one-particle partition function for a GJMS scalar field is

$$\mathcal{Z}_{\text{GJMS}_r}(S^1 \times S^3; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{r-1} (n+1)^2 q^{n+2-r+2k} = \frac{q^{2-r} - q^{2+r}}{(1-q)^4}. \quad (\text{C.7})$$

²⁴Here $R_{mn} = \pm g_{mn}^{(0)}$, where $g_{mn}^{(0)}$ is the metric of a standard 2-sphere and the sign depends on whether (mn) are in the first or second sphere. Thus $\nabla^2 = -\nabla_{S^2}^2 + \nabla_{S^2}^2$ and $R^{mn}\nabla_m\nabla_n = \nabla_{S^2}^2 + \nabla_{S^2}^2$.

This is the same as the partition function that counts descendants of a conformal scalar operator in flat space modulo its equation of motion. In general dimension d , for a GJMS scalar ϕ with canonical dimension $\frac{1}{2}(d-2r)$ and equations of motion $\partial^{2r}\phi = 0$ of complementary dimension $\frac{1}{2}(d+2r)$ we get

$$\mathcal{Z}_{\text{GJMS}_r}(S^1 \times S^{d-1}; q) = \frac{q^{\frac{d}{2}-r} - q^{\frac{d}{2}+r}}{(1-q)^d}. \quad (\text{C.8})$$

C.2 Partition function on AdS_d

The discussion in section 3.1 may be generalized by considering a massive scalar field in AdS_{d+1} with $\Delta_{d+1}^\pm = \frac{1}{2}(d \pm 2r)$ (with $r = 2, 3, \dots$) for which the associated d -dimensional boundary conformal field is the higher derivative conformal scalar with canonical dimension $\Delta_{d+1}^- = \frac{1}{2}(d-2r)$ and the kinetic operator $\hat{\Delta}_{(2r)}$. Here (3.7) is replaced by

$$\mathcal{Z}_{0,r}^\pm(\text{AdS}_{d+1}; q) = \frac{q^{\frac{1}{2}(d \pm 2r)}}{(1-q)^d}. \quad (\text{C.9})$$

On a generic d -dimensional Einstein space $\hat{\Delta}_{(2r)}$ factorizes as follows [52]

$$\hat{\Delta}_{(2r)} = \prod_{k=1}^r \left(-\nabla^2 + \frac{(\frac{d}{2}-k)(\frac{d}{2}+k-1)}{d(d-1)} R \right). \quad (\text{C.10})$$

For AdS_d case we have $R = -d(d-1)$ and using (2.2) then find that for each massive Laplacian factor in (C.10) the associated dimensions are

$$\Delta_{d,k}^+ = \frac{d+2k}{2} - 1, \quad \Delta_{d,k}^- = \frac{d-2k}{2}, \quad k = 1, \dots, r. \quad (\text{C.11})$$

Hence, (3.10) is generalized to (there is no gauge invariance so that $\sigma_d = 0$ and $\tilde{\mathcal{Z}}^- = \mathcal{Z}^-$)

$$\mathcal{Z}_{\text{GJMS}_r}^+(\text{AdS}_d; q) = \sum_{k=1}^r \frac{q^{\Delta_{d,k}^+}}{(1-q)^{d-1}} = \frac{q^{d/2}(1-q^r)}{(1-q)^d}, \quad (\text{C.12})$$

$$\mathcal{Z}_{\text{GJMS}_r}^-(\text{AdS}_d; q) = (-1)^{d+1} \mathcal{Z}_{\text{GJMS}_r}^+(\text{AdS}_d; q^{-1}). \quad (\text{C.13})$$

Comparing (C.9) to (C.12), (C.13) one checks again the relation (3.1), (3.5), (3.1)

$$\mathcal{Z}_{0,r}^-(\text{AdS}_{d+1}; q) - \mathcal{Z}_{0,r}^+(\text{AdS}_{d+1}; q) = \mathcal{Z}_{\text{GJMS}_r}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{GJMS}_r}^+(\text{AdS}_d; q). \quad (\text{C.14})$$

We can also demonstrate the relation (3.2) by using (C.8) and (C.12), (C.13) to check that

$$\mathcal{Z}_{\text{GJMS}_r}(S^1 \times S^{d-1}; q) = \mathcal{Z}_{\text{GJMS}_r}^-(\text{AdS}_d; q) + \mathcal{Z}_{\text{GJMS}_r}^+(\text{AdS}_d; q). \quad (\text{C.15})$$

D Fermionic conformal higher spin fields

The discussion of partition function relations for the bosonic CHS fields may be extended to the 4d fermionic ones (see [20]). These are boundary counterparts for the massless spin s fermionic higher spin fields in AdS_5 to the $SO(2,4)$ representation

$$\text{MHS}_s = \left(s + \frac{5}{2}; \frac{s}{2}, \frac{s+1}{2}\right) + \left(s + \frac{5}{2}; \frac{s+1}{2}, \frac{s}{2}\right), \quad s \equiv s - \frac{1}{2} = 0, 1, 2, \dots \quad (\text{D.1})$$

Here $s = 0$ corresponds to spin $\frac{1}{2}$ fermion, $s = 1$ to conformal gravitino, etc. Recalling that for the massless $SO(2, 4)$ representation $(2 + j_1 + j_2; j_1, j_2)$ we have²⁵

$$\mathcal{Z}_{(2+j_1+j_2;j_1,j_2)}^+(\text{AdS}_5; q) = \frac{q^{2+j_1+j_2}}{(1-q)^4} \left[(2j_1+1)(2j_2+1) - 4q j_1 j_2 \right], \quad (\text{D.2})$$

we obtain (see Eq. (2.17) in [3])

$$\mathcal{Z}_{\text{MHS}_s}^+(\text{AdS}_5; q) = \frac{2(s+1)(s+2)q^{\frac{5}{2}+s} - 2s(s+1)q^{\frac{7}{2}+s}}{(1-q)^4}. \quad (\text{D.3})$$

Applying the "reconstruction" algorithm in Appendix (A) or using the explicit factorized form of the fermionic CHS partition function on S^4 [20] and thus also on AdS_4 one may find that

$$\mathcal{Z}_{\text{CSH}_s}^+(\text{AdS}_4; q) = 2 \frac{(s+1)(q^{\frac{3}{2}} + q^{\frac{5}{2}}) - (s+1)(s+2)q^{\frac{5}{2}+s} + s(s+1)q^{\frac{7}{2}+s}}{(1-q)^4}. \quad (\text{D.4})$$

One then concludes that the relation (3.5) is again satisfied

$$\tilde{\mathcal{Z}}_{\text{MHS}_s}^-(\text{AdS}_5; q) - \mathcal{Z}_{\text{MHS}_s}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{CHS}_s}^-(\text{AdS}_4; q) + \mathcal{Z}_{\text{CHS}_s}^+(\text{AdS}_4; q). \quad (\text{D.5})$$

In addition, the partition function $\mathcal{Z}_{\text{CHS}_s}(S^1 \times S^3; q)$ was found in [3] (see Eq. (2.26) there). Comparing it with (D.4), we conclude that the relation (4.14) holds also for the fermionic CSH_s family, i.e.

$$\mathcal{Z}_{\text{CHS}_s}(S^1 \times S^3; q) = 2 \mathcal{Z}_{\text{CHS}_s}^+(\text{AdS}_4; q). \quad (\text{D.6})$$

E Conformal antisymmetric tensor fields in 4d

The Weyl-covariant Lagrangian for the conformal antisymmetric tensor field $T_{\mu\nu}$ on a generic curved 4d background is [29]

$$\mathcal{L} = (\nabla^\mu T_{\mu\nu})^2 - \frac{1}{4} (\nabla_\mu T_{\rho\sigma})^2 - R_{\mu\nu} T^{\mu\lambda} T_\lambda^\nu + \frac{1}{8} R T_{\mu\nu}^2 + \frac{1}{2} R_{\mu\alpha\nu\beta} T^{\mu\nu} T^{\alpha\beta}. \quad (\text{E.1})$$

This conformal field in flat 4d space corresponds in AdS_5 to a massive spin 1 theory with representation content $\text{HS} = (3; 1, 0) \oplus (3; 0, 1)$ and no gauge invariance. The Lagrangian (E.1) restricted to AdS_4 background gives the kinetic operator that factorizes into vector operators as discussed in [48]. The thermal partition function on $S^1 \times S^3$ may be found in eq. (B.26) of [1]. As a result,

$$\mathcal{Z}_{\text{HS}}^+(\text{AdS}_5; q) = \frac{6q^3}{(1-q)^4}, \quad \mathcal{Z}_{\text{T}}^+(\text{AdS}_4; q) = \frac{6q^2}{(1-q)^3}, \quad \mathcal{Z}_{\text{T}}(S^1 \times S^3; q) = \frac{6q-6q^3}{(1-q)^4}. \quad (\text{E.2})$$

One finds then that (3.5) is satisfied

$$\tilde{\mathcal{Z}}_{\text{HS}}^-(\text{AdS}_5; q) - \mathcal{Z}_{\text{HS}}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{T}}^-(\text{AdS}_4; q) + \mathcal{Z}_{\text{T}}^+(\text{AdS}_4; q), \quad (\text{E.3})$$

²⁵The gauge subtraction in (D.2) is present only for $s > 0$ but since it happens to vanish for $s = 0$ this formula is general.

but there is no analogue of (4.14).

Let us elaborate on the derivation of $\mathcal{Z}_T^+(\text{AdS}_4; q)$ in (E.2). The antisymmetric tensor partition function on S^4 is [48]

$$Z_T(S^4) = \left[\det \hat{\Delta}_{(1,0)}(4) \det \hat{\Delta}_{(0,1)}(4) \right]^{-1/2}, \quad (\text{E.4})$$

where the 2nd order operator $\hat{\Delta}_{(j_1, j_2)}(M^2)$ (cf. (2.1)) acts on a field in an irreducible $SO(1, 3)$ representation (j_1, j_2) (see, e.g., [55, 20]). Similar partition function is found on AdS_4 where the mass term is related the corresponding conformal dimension Δ_4^\pm by the following generalization of (2.2)²⁶

$$\Delta_4^\pm (\Delta_4^\pm - 3) - j_1(j_1 + 1) - j_2(j_2 + 1) = -M^2. \quad (\text{E.5})$$

For $M^2 = 4$ and $(j_1, j_2) = (0, 1)$ or $(1, 0)$ as in (E.4) this gives $\Delta_4^+ = 2$ and $\Delta_4^- = 1$. Therefore, $\mathcal{Z}_T^+(\text{AdS}_4; q) = \frac{6q^2}{(1-q)^3}$, in agreement with (E.2) (the factor 6 is the dimension of $(1, 0) \oplus (0, 1)$ representation).

Note that the partition function in (E.4) leads to the correct value of the conformal a-anomaly coefficient for $T_{\mu\nu}$. Using Eqs. (3.34)-(3.35) of [20], the contribution to the a-anomaly from $\hat{\Delta}_{(j_1, j_2)}(M^2)$ is found to be

$$a_{(j_1, j_2)}(M^2) = \frac{1}{720} (2j_1 + 1)(2j_2 + 1) [10(j_1(j_1 + 1) + j_2(j_2 + 1)) - 15M^4 + 60M^2 - 58], \quad (\text{E.6})$$

so that $a(T) = \hat{a}_{(1,0)}(4) + \hat{a}_{(0,1)}(4) = -\frac{19}{60}$, in agreement with [48, 3].²⁷ Indeed, the general form of (4.6) is [1]

$$a(\Delta_5; j_1, j_2) = \frac{1}{720} (2j_1 + 1)(2j_2 + 1)(\Delta_5 - 2) \times \left[-3(\Delta_5 - 2)^4 + 10(j_1^2 + j_2^2 + j_1 + j_2 + \frac{1}{2})(\Delta_5 - 2)^2 - 15(j_1 - j_2)^2(j_1 + j_2 + 1)^2 \right], \quad (\text{E.7})$$

and we again get $a(3; 1, 0) + a(3; 0, 1) = -\frac{19}{60}$.

One can repeat the above discussion for a conformal 4d field T_p transforming in the $(p, 0) \oplus (0, p)$ representation of the $SO(1, 3)$.²⁸ We may start in AdS_5 with a 5d field in $(\Delta_5; p, 0) \oplus (\Delta_5; 0, p)$ representation (to be denoted as HS_p). It should correspond to a conformal field in AdS_4 with the canonical dimension $4 - \Delta_5$ and thus with the kinetic term $T_p \square^{\Delta_5-2} T_p + \dots$. The correspondence rule (4.1) here reads as

$$(\Delta_5; p, 0) \oplus (\Delta_5; 0, p) \rightarrow Z_{T_p}(\text{AdS}_4) = \prod_{k=1}^{\Delta_5-2} \left[\det \hat{\Delta}_{(p,0) \oplus (0,p)}(2 + p(p+1) - k(k-1)) \right]^{-1/2} \quad (\text{E.8})$$

²⁶The explicit dependence on $j_{1,2}$ through the Casimir of $SO(4)$ comes from the particular definition of the operator, see Eq. (3.5) of [20].

²⁷As explained in [56], it is also possible to express $T_{\mu\nu}$ in terms of two spin 1 vector fields and thus write the partition function in the form (see Eq.(5) of [48], cf. (4.25)) $Z_T(S^4) = C [\det \hat{\Delta}_{1\perp}(3)]^{-1}$, where C accounts for the zero mode contributions (cf. discussion after Eq.(6.18) in [29]). This zero mode factor is essential to reproduce the correct value for the a-anomaly giving extra $-\frac{1}{2}$ shift: using (4.7) to find $\hat{a}_{1\perp}(3)$ we get $\hat{a}(T) = 2\hat{a}_{1\perp}(3) - \frac{1}{2} = -\frac{19}{60}$.

²⁸Fields transforming in the $(p, 0) \oplus (0, p)$ representation are Weyl-like tensors (see, e.g., [57]). They may be represented as rank $2p$ tensors $T_{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_p \nu_p}$ antisymmetric in each pair $\mu_i \nu_i$, totally symmetric with respect to the exchange of the pairs $(\mu_i \nu_i)$ and $(\mu_j \nu_j)$, traceless, and obeying a generalized "Bianchi identity" $T_{\dots[\mu\nu\rho]} = 0$.

Using (E.6) and (E.7) one finds that the equality of the a-anomaly coefficients implied by (E.8) indeed holds

$$a(\Delta_5; p, 0) + a(\Delta_5; 0, p) = 2 \sum_{k=1}^{\Delta_5-1} a_{(p,0)}(2 + p(p+1) - k(k-1)) . \quad (\text{E.9})$$

From (E.5) we find that the dimension corresponding according to (2.2) to the k -th operator in the r.h.s. of (E.8) is $\Delta_4^{(k)} = k + 1$, so that from (2.3) we get

$$\begin{aligned} \mathcal{Z}_{\text{HS}_p}^+(\text{AdS}_5; q) &= \frac{2(2p+1)q^{\Delta_5}}{(1-q)^4}, \\ \mathcal{Z}_{\text{T}_p}^+(\text{AdS}_4; q) &= \frac{2(2p+1)}{(1-q)^3} \sum_{k=1}^{\Delta_5-2} q^{k+1} = \frac{2(2p+1)}{(1-q)^4} (q^2 - q^{\Delta_5}) . \end{aligned} \quad (\text{E.10})$$

Thus once again we get the relation (3.5)

$$\tilde{\mathcal{Z}}_{\text{HS}_p}^-(\text{AdS}_5; q) - \mathcal{Z}_{\text{HS}_p}^+(\text{AdS}_5; q) = \tilde{\mathcal{Z}}_{\text{T}_p}^-(\text{AdS}_4; q) + \mathcal{Z}_{\text{T}_p}^+(\text{AdS}_4; q) . \quad (\text{E.11})$$

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